RESEARCH SUMMARY

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1. Introduction

My research area is commutative algebra, which concerns commutative rings and modules over them. Much of my work has involved studying these classical commutative algebra objects from a combinatorial perspective, which has been very fruitful. Combinatorial commutative algebra is a relatively new field, beginning in the 1970s with the work of Stanley and Hochster on simplicial homology and squarefree monomial ideals. The field has produced many notable areas of study, including the development of the theory of Stanley-Reisner rings and simplicial complexes, the connections between graphs and their edge ideals, and semigroup rings and toric varieties.

One fundamental problem in commutative algebra concerns the behavior of powers of an ideal I in a commutative Noetherian ring R. The Rees algebra is a classical commutative algebra construction whose properties give a lot of information about these powers, especially concerning their integral closures. When a second ideal is also considered, we would like to understand how powers of I and J relate to one another. My research concerns an analog of the Rees algebra called the intersection algebra, an object that came from commutative algebra with connections to algebraic geometry, combinatorics, number theory, and optimization. Much like the Rees algebra does for one ideal, the intersection algebra captures the information required to understand these relationships for two ideals. This algebra involves many of my mathematical interests, specifically combinatorial commutative algebra and designing and implementing computational algorithms, especially in the open source platform Macaulay2.

Studying this algebra has also lead to some other interesting examples of finitely generated algebras which have similar properties to the intersection algebra, which we are calling fan algebras. All of these examples come from certain semigroups in $\mathbb{N}^m$. One of my current interests is in uniting these examples under one cohesive theory, and exploring their properties.

2. Definitions, Motivation, and History

Definition 2.1. *The intersection algebra* of two ideals I and J in a ring R is defined to be

$$\mathcal{B}_R(I,J) = \bigoplus_{r,s\in\mathbb{N}} (I^r \cap J^s)\, u^r v^s.$$

Additionally, this definition can be extended in a natural way to an arbitrary number of ideals. Not much is known about the intersection algebra in general. It was studied by J. B. Fields in [1, 2], where he showed many interesting properties. Specifically, in the case where I and J are monomial ideals in the power series ring, he proved that the intersection algebra is always finitely generated. He also related the finite generation to polynomial behavior of a certain function based on the length of Tors of related modules. Notably, he also showed that the intersection algebra is not always finitely generated.

The intersection algebra can be seen as a natural extension of a useful object in commutative algebra called the Rees algebra. In a ring R, with I an ideal and t an indeterminate over R, the algebra $R[It] = \bigoplus_n I^n t^n \subset R[t]$ is called the Rees algebra of the ideal I. The Rees algebra has been extensively studied, notably in [3] and [4], and its connections to integral closure of ideals and blowups in algebraic geometry are well known. In certain cases, the intersection algebra reduces to the double Rees algebra

$$R[Iu, Jv] = \bigoplus_{(r,s)\in\mathbb{N}^2} I^r J^s u^r v^s,$$

but in general is much more complicated. These complications arise because, in general, $I^r \cap J^s$ is much more difficult to predict than $I^r J^s$.

Studying this algebra also gives information on how powers of two ideals interact with each other. Nagata and Samuel first considered this interaction in [5] and [6], and asked which powers of an ideal land in which powers of a different ideal. Under certain conditions on two ideals I and J, one can define $v_I(J, m)$ to be the largest integer n such that $J^m \subseteq I^n$ and $w_J(I, n)$ to be the smallest m such that $J^m \subseteq I^n$. The two sequences $\{v_I(J, m)/m\}_m$ and $\{w_J(I, n)/n\}_n$ have limits $l_I(J)$ and $L_J(I)$, respectively, and Samuel showed that these limits are always rational. They also developed an equivalence relation based on these powers that leads to results on integral dependence and cancellation laws for ideals.

The intersection algebra has also been studied in [7], where the authors showed that that the Noetherianity of this algebra implies certain uniform behaviors for the powers of the ideals I and J. Specifically, if the intersection algebra of I and J is Noetherian, then there exists a positive integer t such that $v_I(J, m + t) = v_I(J, m) + v_I(J, t)$ for all $m \geq t$.

These ideas have recently been studied in positive characteristic. In [8], Huneke et. al. have extended these limits to rings of positive characteristic, and connected it to the concept of tight closure of an ideal. Given a Noetherian ring of characteristic p, let I and J be ideals with $I \subset \sqrt{J}$. Then for every positive integer e, define ν_f to be the largest power a such that f^a is not in $m^{[p^e]}$, where the bracket power is defined to be the Frobenius power. Then the limit of the sequence ν_f/p^e as e goes to infinity is defined to be the F-pure threshold, or fpt of f. It has been shown that, in characteristic 0, the F-pure threshold corresponds to the log canonical threshold. Further, the fpt gives information about when I is contained in the tight closure of J.

Many of my results come from the underlying semigroup ring structure of this algebra in certain cases. The following result uses semigroups to prove the finite generation of an intersection algebra in a more general case than Fields' result. I demonstrated the case of two principal ideals in a UFD in [9], and the more general case (with an arbitrary number of ideals) will appear in an upcoming paper.

Theorem 2.2. *Given a polynomial ring R and two monomial ideals I and J, $\mathcal{B}$ is finitely generated.*

The proof of theorem shows the connection between the intersection algebra and combinatorial commutative algebra. This algebra has a natural lattice structure, described by a fan of pointed rational cones inside $\mathbb{N}^2$. The crux of my proof of the finite generation of $\mathcal{B}$ lies in showing that these cones are finitely generated by a set called a Hilbert basis, and those naturally correspond to generators for $\mathcal{B}$ as an R-algebra. Since addition in the semigroup corresponds to multiplication in $\mathcal{B}$, a generating set for the semigroup gives a generating set for the algebra, and therefore $\mathcal{B}$ is finitely generated as an algebra over R. In fact, $\mathcal{B}$ is a semigroup ring. Note that these generators can easily be produced algorithmically, and I have written a Macaulay2 function to do so for the case of two monomial ideals.

Defining this semigroup explicitly allows us to calculate many more interesting properties of the algebra. We can also obtain a presentation of $\mathcal{B}$ as a quotient of a larger polynomial ring by computing the lattice ideal associated to the semigroup using methods shown in [10]. In examples, we have seen an interesting connection between intersection algebras and determinantal ideals of matrices of indeterminates that warrants further study.

Studying the intersection algebra has lead me to consider a large class of bigraded algebras that display interesting combinatorial properties. The underlying structure of these algebras also relies on semigroups in $\mathbb{N}^2$. Using the same Hilbert basis construction, we have shown that a different algebra, called a fan algebra, is finitely generated.

Theorem 2.3. *Given a Noetherian ring R with ideals $I_1, \dots, I_n$, a fan $\Sigma_{\mathbf{a},\mathbf{b}}$, and functions $f_1(r,s), \dots, f_n(r,s)$ that are subadditive and nonnegative on all of $\mathbb{N}^2$, and linear on each cone in $\Sigma_{\mathbf{a},\mathbf{b}}$, then the algebra*

$$\mathcal{B} = \bigoplus_{r,s} I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} u^r v^s$$

is finitely generated.

This definition is easier to see with an example:

Example 2.4. Let $\Sigma_{\mathbf{a},\mathbf{b}}$ is the fan defined by two cones

$$C_0 = \{\lambda_1(0,1) + \lambda_2(1,1) | \lambda_i \in \mathbb{R}_{\geq 0}\}$$
$$C_1 = \{\lambda_1(1,1) + \lambda_2(1,0) | \lambda_i \in \mathbb{R}_{\geq 0}\}.$$

Also let

$$f = \begin{cases} g_0(r,s) = r + 2s & \text{if } (r,s) \in C_0 \cap \mathbb{N}^2 \\ g_1(r,s) = 2r + s & \text{if } (r,s) \in C_1 \cap \mathbb{N}^2 \end{cases}$$

Then f is clearly nonnegative and linear on both C_0 and C_1, and it can easily be shown that the function is also subadditive on all of $\mathbb{N}^2$. We call a function that obeys these properties fan-linear. Then, for an ideal $I = (x_1, \dots, x_m)$ in a Noetherian ring R, we can define a fan algebra

$$\mathcal{B} = \bigoplus_{r,s} I^{f(r,s)} u^r v^s,$$

and it can be shown, using a Hilbert basis argument, that $\mathcal{B}$ in the above example is finitely generated over R by the set

$$\{x_i u, x_i v, x_i^3 uv | i = 1, \dots, m\}.$$

This fan algebra is also important because certain cases of the intersection algebra can be reduced to algebras of this form. For instance, in an $\mathbb{N}$-graded Noetherian ring R, the intersection algebra of a principal ideal $I = (f)$, where f has degree a, and a maximal ideal $\mathfrak{m}$ can be reduced to the algebra $\bigoplus_{r,s} \mathfrak{m}^{s-ra}$. So certain kinds of intersection algebras can be shown to be finitely generated as a result of the finite generation of these product algebras.

3. Current Work

My latest paper [11], which is joint work with my advisor, Florian Enescu, comes at the intersection algebra from a different perspective, that of systems of diophantine equations. In [12], Stanley developed an interpretation of semigroups as the set of solutions to systems of such equations. His definition is below.

Definition 3.1. Let Φ be an $r \times n$ $\mathbb{Z}$-matrix, $r \leq n$, and rank $\Phi = r$. Define

$$E_\Phi := \{\boldsymbol{\beta} \in \mathbb{N}^n | \Phi\boldsymbol{\beta} = 0\} \subset \mathrm{Ker}(\Phi).$$

Then E_Φ is clearly a subsemigroup of $\mathbb{N}^n$.
Let $R_\Phi := kE_\Phi$, the semigroup algebra of E_Φ over k. We identify $\boldsymbol{\beta} \in E_\Phi$ with $x^{\boldsymbol{\beta}}$, so that $R_\Phi \subseteq k[x_1, \dots, x_n]$ as a subalgebra graded by monomials.

To translate our problem into this language, we must describe all monomials in $\mathcal{B}$ as having exponents that are solutions to a system of equations. We restrict ourselves to principal ideals in the monomial ring in n variables, so $I = (x_1^{a_1} \cdots x_n^{a_n})$ and $J = (x_1^{b_1} \cdots x_n^{b_n})$. Since any monomial in $\mathcal{B}$ must have the form $(\prod_i x_i^{m_i})u^r v^s$, with $m_i \geq \max(a_i r, b_i s)$, there exists an $h_i, k_i \in \mathbb{N}$ such that the exponent vector of any monomial must satisfy

$$m_i = a_i r + h_i = b_i s + k_i, i = 1, \dots, n.$$

Now let

$$\Phi = \Phi_{a,b} = \begin{pmatrix} a_1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & b_1 & 0 & 1 & -1 & 0 & 0 & 0 & \dots & 0 \\ a_2 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & \dots & 0 \\ 0 & b_2 & 0 & 0 & 0 & 0 & 1 & -1 & \dots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \end{pmatrix},$$

and $\boldsymbol{\beta} = (r, s, h_1, k_1, m_1, \dots, h_n, k_n, m_n) \in \mathbb{N}^{3n+2}$.

Then $E_\Phi = E_{\Phi_{a,b}} = \{\boldsymbol{\beta} \in \mathbb{N}^{3n+2} | \Phi\boldsymbol{\beta} = 0\}$. So E_Φ is a subsemigroup in $\mathbb{N}^{3n+2}$, and to recover our semigroup Q where $\mathcal{B} = k[Q]$, we project E_Φ onto $\mathbb{N}^{n+2}$ by "forgetting" all h_i and k_i. This establishes an isomorphism between Q and E_Φ, and so $\mathcal{B}$ and R_Φ are isomorphic as well.

Reformulating $\mathcal{B}$ in terms of this matrix allows us to easily prove some more properties of this algebra, namely its Hilbert series and canonical ideal. We have also shown that the only case of an intersection algebra of two principal monomial ideals that is Gorenstein is $I = J = (x^a)$ for some number a.

We are also investigating our intersection algebras from yet another angle, this time as algebras arising from toric varieties. Toric varieties are well studied, and our algebras are always coordinate rings of such varieties. We have begun to use already developed methods by several authors to compute the f-pure threshold of our algebras in the case when the coefficient ring is of positive characteristic.

4. Computational Algebra

Another of my research interests is computational commutative algebra, specifically using Macaulay2. I've had the opportunity to attend two workshops with a large M2 component, beginning at the MSRI summer graduate workshop in 2011. There, I worked closely with Irena Swanson and other students to begin the development of a new package to compute integral closure of a ring via a process developed by Seidenberg and Stolzenberg in [13–15].

Later, I attended the M2 workshop at Wake Forest University, where I assisted Karl Schwede and Emily Witt with constructing a package for computations in positive characteristic. One of the main functions in the package involves estimating the f-pure threshold of a polynomial and the maximal ideal of a polynomial ring, and calculating it exactly in certain cases. The package uses algorithms from various papers by Schwede and others, notably in [16].

I have also used M2 extensively in my own research, most notably by implementing the algorithm discussed above to calculate generating sets for intersection algebras. The algorithm originally appeared in [9], and an updated version that covers more cases will appear in an upcoming paper.

Macaulay2 presents many exciting opportunities for undergraduate researchers. The system is well documented and open source, with an active and supportive community, and it provides a nice framework to introduce students to mathematical programming. It allows students to easily generate dozens of examples and begin the process of forming conjectures. The online community is very active and supportive, and many undergrads and early graduate students have published the results they obtained in these workshop

5. Future Work

Despite the relative simplicity of the definition of the intersection algebra, there are still many open questions. While I know that some of my results can be generalized from the principal ideal case to that of any monomial ideals, I do not yet have a full description of the semigroup as solutions to systems of linear equations. Moreover, there is much to be done in extending from polynomial rings to arbitrary rings. I believe many of these questions can be fruitfully explored by undergraduate researchers, especially with the assistance of Macaulay2.

Other future areas of study concern the connections between these algebras and toric varieties. The natural cone structure of the algebra generators, and the connections to semigroup algebras in the polynomial case, suggests that this will be an interesting topic for exploration. I believe many of the semigroups I study are normal, and give rise to normal semigroup rings, and I want to examine what other results this fact can produce. Also, in the polynomial case, the lattice ideals of the semigroups can be explicitly computed, so the next natural step would be to examine free resolutions of those lattice ideals.

I am also interested in studying the interplay between integral closure and intersection algebras. Since the integral closure of a monomial ideal is monomial, and monomial ideals have finite intersection algebra, we know that $\bar{\mathcal{B}} = \bigoplus_{r,s} \bar{I}^r \cap \bar{J}^s u^r v^s$ must be finitely generated, but we have not investigated the structure of this algebra. And $\tilde{\mathcal{B}} = \bigoplus_{r,s} \overline{I^r} \cap \overline{J^s} u^r v^s$ is even more complex. Given the relationships between the Rees algebra and integral closure, I suspect this to be an interesting topic for exploration. Also, since integral closures of monomial ideals can be easily computed using Newton polytopes, this is another excellent access point for undergraduate students.

I would also like to continue my work creating packages for Macaulay2. I have participated in two Macaulay2 workshops where significant progress was made on creating new packages, and my collaborators are interested in continuing the work. Especially in the case of computations in characteristic p, little has been done to implement various algorithms in the language, and there is a lot of room for expansion.

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